Pergamon

Appl. Math. Lett. Vol. 7, No. 2, pp. 77–79, 1994

0893-9659/94 $6.00 + 0.00

0893-9659(94)E0016-5

The Fundamental Theorem of Game Theory Revisited

M. J. PANIK
Department of Economics
Barney School of Business and Public Administration
University of Hartford, West Hartford, CT 06117, U.S.A.

(Received April 1993; accepted September 1993)

Abstract—This paper offers an alternative proof of the so-called fundamental theorem of the theory of games or the minimax theorem. The proof uses a particularization of Farkas' theorem involving the expression of one vector as a convex combination of a set of vectors. It is demonstrated that the minimax theorem holds as a consequence of this specialization of Farkas' theorem of the alternative.

Keywords—Game theory, Minimax theorem, Farkas' theorem, Zero-sum games.

1. INTRODUCTION

The fundamental or minimax theorem of two-person zero-sum games was first developed by von Neumann [1] in 1928 using Brouwer's fixed point theorem. Since then this important result has received a considerable amount of attention, with a great deal of effort being devoted to alternative methods of proof (see, for instance: [2–12], among others).

In what follows, we shall suggest yet another proof of the fundamental theorem which essentially "falls out" of a particular extension of Farkas' theorem to a convex combination of points. Given the nature of the proof presented below, it may be broadly classified as one involving the use of a strong separation theorem for convex sets since Farkas' theorem can be construed as a consequence of the same.

2. THE FUNDAMENTAL THEOREM OF GAME THEORY

Let $B = [b_{ij}]$, $i = 1, \dots, m$; $j = 1, \dots, n$, represent the payoff matrix for a two-person zero-sum game (here row i of B gives the expected payoffs to Player 1 if he uses strategy i, $i = 1, \dots, m$, while the columns of B reflect the strategies of Player 2). Furthermore, mixed strategies for Players 1, 2 are defined as $u \geqq 0$, $1'u = 1$ and $v \geqq 0, 1'v = 1$ respectively, where $u_i\,(v_j)$ is the probability of choosing pure strategy $i(j)$.

If Player 1 employs the mixed strategy u and Player 2 likewise uses v, then the expected winnings for Player 1 are $E(u,v) = u'Bv$. Player 1 desires to find a u which maximizes his minimum expected gain so that his expected winnings are at least

$$E_1^* = \max_u \min_v E(u,v);$$

and Player 2 endeavors to hold Player 1's expected gain to no more than

$$E_2^* = \min_v \max_u E(u,v).$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

The solution to a two-person zero-sum game involves finding a set of optimal mixed strategies u^*, v^* which yield the optimal value of the game $E^* = E_1^* = E_2^*$. In this regard, the "fundamental (minimax) theorem for two-person zero-sum games" informs us that every matrix game has a solution in mixed strategies, i.e., there always exist u^*, v^* for which

$$\begin{aligned} &E_1^* = \max_u \min_v E(u,v) = E^* = \min_v \max_u E(u,v) = E_2^*, \\ &u^* \geqq 0, 1'u^* = 1 \text{ and } v^* \geqq 0,\ 1'v^* = 1. \end{aligned} \tag{1}$$

3. PROOF OF THE FUNDAMENTAL THEOREM

To set the stage for a proof of the fundamental theorem of two-person zero-sum games let us first consider the following extension of Farkas' theorem [13]:

FARKAS' THEOREM UNDER CONVEX COMBINATION. *For an* $(m \times n)$ *matrix* A *and a vector* $b \epsilon \Re^m$, *either*

(I) $AX = b, X \geqq 0, 1'X = 1$ *has a solution* $X \in \Re^n$ *or,*

(II) $A'Y + 1y \geqq 0, (b'Y, y) \leq 0'$ *has a solution* $y \in \Re$, $Y \in \Re^m$

but never both.

Next, if we again address the problem from the viewpoint of Player 1, we seek to determine a vector u which satisfies

$$B'u \geqq E_1 1, \quad u \geqq 0, \quad 1'u = 1, \tag{2}$$

i.e., Player 1 will never expect to win more than the largest value of E_1 for which there exists a u satisfying (2). Similarly, Player 2 seeks to find a vector v which will render the smallest E_2 satisfying

$$Bv \leqq E_2 1, \quad v \geqq 0, \quad 1'v = 1. \tag{3}$$

Based upon these considerations we may rewrite (2) as

$$B'u - S = E_1 1, u \geqq 0, S \geqq 0, 1'u = 1 \ \text{ or } \ [B', -I_n]\begin{bmatrix} u \\ S \end{bmatrix} = E_1 1,\ \begin{bmatrix} u \\ S \end{bmatrix} \geqq 0, [1', 0']\begin{bmatrix} u \\ S \end{bmatrix} = 1. \tag{2.1}$$

Then according to the preceding theorem, either (2.1) has a solution $[u', S'] \geqq 0'$ for $u \in \Re^m, S \in \Re^n$ or, according to (II), there exists a $y \in \Re, Y \in \Re^n$ such that

$$\begin{bmatrix} B \\ -I_n \end{bmatrix} Y + \begin{bmatrix} 1 \\ 0 \end{bmatrix} y \geqq 0, \quad E_1 1'Y \leqq 0, \quad y < 0. \tag{4}$$

Clearly, (4) is equivalent to

$$BY \geqq -1y > 0, \quad Y \leqq 0, \quad 1'Y \leqq 0, y < 0 \tag{4.1}$$

(since we can assume that $E_1 > 0$). Since (4.1) admits an obvious inconsistency (Figure 1), it follows that (2.1) has a solution (call it $u^* \geqq 0$).

In this regard,

$$(u^*)'Bv \leqq E_2(u^*)'1 = E_2.$$

Similarly,

$$v'B'u^* \geqq E_1^* v'1 = E_1^*$$

so that

$$E_2 \geqq (u^*)'Bv \geqq E_1^*.$$

And since every matrix game has at most one value, there exists a $v = v^*$ such that $E_1^* = E_2^*$ and thus, the proof is complete.

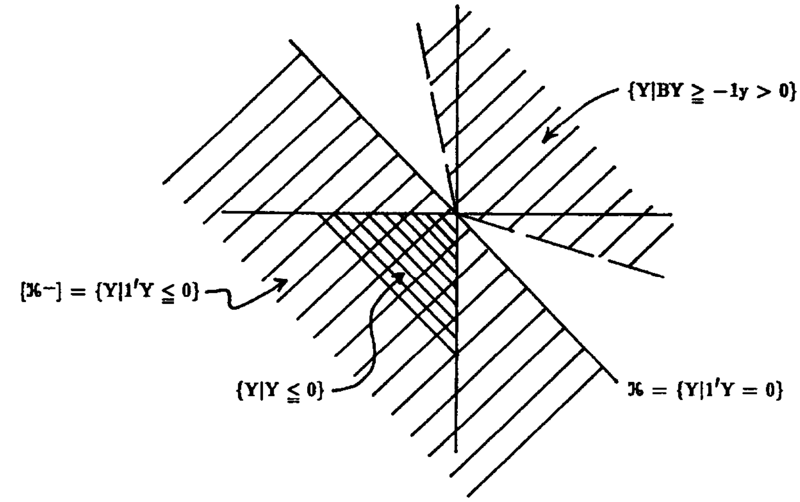

Figure 1.

REFERENCES

1. J. von Neumann, Zur Theorie der Gesellschaftsspiele, *Mathematiscle Annalem* **100**, 295–320 (1978).
2. S. Kakutani, A generalization of Brouwer's fixed-point theorem, *Duke Mathematical Journal* **8**, 451–459 (1941).
3. J. Nash, *Proceedings of the National Academy of Sciences, U.S.A.* **36**, 48–49 (1950).
4. J. Ville, Note Sur la Theorie Generale des Jeux on Intervieut l'Habilite des Jouer, *Traite du Calcul des Probabilites et deses Applications*, (Edited by E. Borel *et al.*), Tome IV, Fascicule II, pp. 105–113, Paris, (1938).
5. J. von Neumann and O. Morgenstern, *Theory of Games and Economic Behavior*, 2nd ed., Princeton University Press, Princeton, (1947).
6. D. Gale, *The Theory of Linear Economic Models*, McGraw-Hill Book Co. Inc., New York, (1960).
7. D. Gale, D, H. Kuhn and A.Tucker, On symmetric games, In Contributions to the Theory of Games, I, (Edited by H. Kuhn and A. Tucker) Annals of Mathematics Studies, Vol. 24, pp. 81–88, Princeton University Press, Princeton, (1950).
8. H. Kuhn and A. Tucker, Editors, Contributions to the Theory of Games, I, *Annals of Mathematics Studies*, Vol. 24, Princeton University Press, Princeton, (1950).
9. G. Dantzig, Constructive proof of the min-max theorem, *Pacific Journal of Mathematics*, 25–33 (1956).
10. L.H. Loomis, On a theorem of von Neumann, *Proceedings of the National Academy of Sciences, U.S.A.* **32**, 213–215 (1946).
11. H. Weyl, Elementary proof of a minimax theorem due to von Neumann, *Annals of Mathematics Studies*, Vol. 24, pp. 19–25, Princeton University Press, Princeton, (1950).
12. L. Shapley and R. Snow, Basic solutions of discrete games, *Annals of Mathematics Studies*, Vol. 24, pp. 27–35, Princeton University Press, Princeton, (1950).
13. M. Panik, Farkas's theorem under convex combination, *Appl. Math. Lett.* **3** (3), 89–90 (1990).